Pergamon

*J. Appl. Maths Mechs*, Vol. 61, No. 4, pp. 587–600, 1997

0021–8928/97 $24.00+0.00

PII: S0021–8928(97)00076-2

# SOLITARY AND GENERALIZED SOLITARY WAVES IN DISPERSIVE MEDIA†

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(*Received* 28 *March* 1995)

The existence of plane solitary and generalized solitary waves in slightly dispersive media is proved. The wave processes in such media are described by systems of invertible partial differential equations which are subject to natural conditions, formulated in the paper. The proof is carried out by reducing the order of the corresponding dynamical system describing travelling waves and a subsequent investigation of the flow in the central manifold of the system. © 1997 Elsevier Science Ltd. All rights reserved.

The sufficient conditions for solitary waves to exist were formulated in [1] for a wide class of scalar equations of the first and second order in time, which are invariant under time inversion and inversion of the spatial variable. In this paper we extend the results obtained in [1] to the case of invertible systems of equations of arbitrary order. Generalized solitary waves, the conditions for the existence of which are considered in this paper, are a product of the non-linear resonance of a solitary wave and a periodic wave of considerably smaller amplitude than the solitary component. The ordinary solitary wave in such a system ceases to be a travelling wave and decays very slowly with time.

To investigate questions of existence we will use local methods, i.e. the question concerns the existence of families of solitary waves whose amplitude is fairly small. Such solitary waves arise in physical problems as a bifurcation from a state of rest with zero wave number. A situation when the dispersion relation reveals resonance between a wave travelling with a phase velocity for zero wave number and a linear wave corresponding to a non-zero wave number is also typical. In this case solitary waves of a new type are possible, namely, generalized solitary waves, which carry a periodic ripple which does not decay at infinity (see, for example, [2]).

To prove the existence of solutions of the solitary and generalized solitary-wave type for systems of the form (1.1), which satisfies the state of rest $\mathbf{w} = 0$, we use the method of reducing dynamical systems, describing travelling waves, to systems of lower order on a central manifold. The latter is an integral manifold in the phase space of the system, where there are bounded solutions, which do not leave a fairly small neighbourhood of zero [3, 4]. Further investigation of the reduced system reduces to a study of integrable systems, which approximate to this system as accurately as desired in the neighbourhood of zero. The approximating systems are a normal form of equations on the central manifold. Two types of normal forms of the systems are used, which describe solitary and generalized solitary waves [2].

## 1. FORMULATION OF THE PROBLEM. THE EXISTENCE OF SOLITARY-WAVE TYPE SOLUTIONS

Consider a system of $n$th order equations describing the propagation of plane waves in dispersive media

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)\mathbf{w}+\mathbf{F}\left(\mathbf{w},\frac{\partial^i}{\partial x^i}\mathbf{w},\frac{\partial^i}{\partial x^i}\frac{\partial}{\partial t}\mathbf{w}\right)=0,\quad i\leqslant r,\ j\leqslant r-1,\ \mathbf{w}\in\mathbf{R}^r \tag{1.1}$$

$$L\left(\frac{\partial}{\partial x},\frac{\partial}{\partial t}\right)=\sum_{i=1}^{r}A_i\frac{\partial^i}{\partial x^i}+\sum_{j=1}^{r-1}B_j\frac{\partial^j}{\partial x^j}\frac{\partial}{\partial t}+C\frac{\partial}{\partial t}$$

where $A_i$, $B_j$ and $C$ are constant $n\times n$ matrices. The vector $\mathbf{F}$ depends non-linearly on the arguments denoted in (1.1). In addition, we will assume that system (1.1) is in variant under the simultaneous

†*Prikl. Mat. Mekh.* Vol. 61, No. 4, pp. 606–620, 1997.

inversion $t \to -t$ and $x \to -x$. We will henceforth call this invariance invertibility. The dispersion relation for (1.1) is obtained from the equation $L\mathbf{w} = 0$ where $\mathbf{w} = \mathbf{c}\exp\{i(kx - \omega t)\}$, while $c$ is a constant vector.

For Eq. (1.1) the dispersion equation has $n$ branches and can be written in the form

$$S = \prod_{i=1}^{n} (\omega - g_i(k)) = 0 \tag{1.2}$$

We will put $\omega_i = g_i(k)$ and $c_i = \lim_{k\to 0}\omega_i/k$—the phase velocity of infinitely long waves for the $i$th branch.

*Proposition* 1 (on the dispersion relation). A phase velocity $c_{i_0} \neq c_i$ exists for any number $i \neq i_0$ such that the order of tangency of the straight line $c_{i_0}k$ of the graph of branch $\omega_{i_0}$ for $k = 0$ is equal to 2, and this straight line does not intersect the graph of any branch of the dispersion equation (1.2) anywhere apart from the point $k = 0$.

We will further denote the corresponding velocity $c_{i_0}$ by $c$. We will be interested in travelling-wave type solutions of Eqs (1.1), when the unknown function $\mathbf{w}$ depends only on $\xi = x - Vt$, where $V$ is the velocity of the travelling wave. The equation describing travelling waves is obtained from (1.1) by making the replacements $\partial/\partial x \to \partial/\partial\xi$, $\partial/\partial t \to -V\partial/\partial\xi$.

In the weak-dispersion case considered $\lim_{k\to 0} g_i(k)/k < \infty$. After changing to the system describing travelling waves and the replacement of $t$ and $x$ related to this change, in the dispersion relation (1.2) the frequency $\omega$ is replaced by $Vk$. By virtue of the corresponding properties of the functions $g_i(k)$, after taking the common factor $k^n$ outside the produce sign in (1.2), the limit as $k \to 0$ of the expression which remains under the product sign is finite. Moreover, if Proposition 1 is satisfied, this expression will obviously have a double zero root when $V = c$.

It follows from the above that the equation

$$\mathscr{L}(\partial/\partial\xi, -V\partial/\partial\xi)\exp(ik\xi) = 0$$

where $\partial\mathscr{L}/\partial\xi = L$, when $V = c$ has a zero root only of the second order and single integration eliminates the zero of order $n$ in the equation obtained from (1.2) after the above replacements.

*Proposition* 2 (on the non-linearity of the vector $\mathbf{F}$). System (1.1) for travelling waves can be rewritten in the form

$$G(V,\mathbf{w})\left\{L\left(\frac{\partial}{\partial\xi}, -V\frac{\partial}{\partial\xi}\right)\mathbf{w} + \frac{\partial}{\partial\xi}f\left(\mathbf{w}, \frac{\partial^k}{\partial\xi^k}\mathbf{w}\right)\right\} = 0, \quad k \leqslant n-1 \tag{1.3}$$

where $G(V, \mathbf{w})$ is non-degenerate in the neighbourhood of $\mathbf{w} = 0$ and $Vi = \mathrm{c}$ is an $n \times n$ matrix. System of equations (1.3) for solutions $\mathbf{w}$, decreasing at infinity, is therefore equivalent to the system

$$\mathscr{L}\mathbf{w} + \mathbf{f} = 0 \tag{1.4}$$

*Proposition* 3. The system of equations (1.4) is solvable for the higher derivatives in the neighbourhood of $\mathbf{w} = 0$ and $V = c$.

It immediately follows from Proposition 3 that system (1.4) can be rewritten in the form of the dynamical system

$$\dot{\mathbf{v}} = A(c)\mathbf{v} + \mathscr{L}(\mu, \mathbf{v}), \quad \mu = V - c, \quad \mathbf{v} \in \mathbf{R}^m \tag{1.5}$$

where $A(c) = A$ is a constant $n \times n$ matrix, and the dot denotes differentiation with respect to "time", the role of which here is played by the spatial variable $-\infty < \xi < \infty$. The non-linearity of $\mathscr{F}$ comprises the term $A(V) - A(c)$ so that $\mathscr{F}(0, 0) = 0$ and $\partial\mathscr{F}(0, 0)/\partial\mathbf{v} = 0$.

The invertibility of system (1.1) and the related invertibility of the dynamical system (1.5) (invariance under the replacement $\xi \to -\xi$) denotes that a real diagonal $m \times m$ matrix $R$: $\mathbf{R}^m \to \mathbf{R}^m$ exists such that $R^2 = 1$ and $R$ anticommutes with the left-hand side of (1.5), i.e. $AR = -RA$ and $\mathscr{F}(\mu, R\mathbf{v}) = -R\mathscr{F}(\mu, \mathbf{v})$.

For Proposition 1 to hold, the only eigenvalue of the matrix $A$ which lies on the imaginary axis will be a zero of multiplicity 2. In the case of a common situation, the matrix $A$ has a single eigenvector $\phi_0$

and one associated vector $\boldsymbol{\phi}_1$: $A\boldsymbol{\phi}_0 = 0, A\boldsymbol{\phi}_1 = \phi_0$. The phase space $\mathbf{R}^m$ is therefore the direct sum of $E_0$ and $E_h$, where $E_0$ is the central invariant space of the operator $A$ (of dimension 2 in this case), while $E_h$ is a hyperbolic invariant subspace $A$.

The solution of system (1.5) can be represented in the form $\mathbf{v} = \mathbf{v}_0 \otimes \mathbf{v}_h$, where $\mathbf{v}_0 \in E_0$ while $\mathbf{v}_h \in E_h$. System (1.5) can be projected onto the half-spaces $E_0$ and $E_h$

$$\dot{\mathbf{v}}_0 = A_0\mathbf{v}_0 + \mathcal{F}_0(\mu, \mathbf{v}_0 + \mathbf{v}_h), \quad \dot{\mathbf{v}}_h = A_1\mathbf{v}_h + \mathcal{F}_1(\mu, \mathbf{v}_0 + \mathbf{v}_h) \tag{1.6}$$

($A_0 = A|_{E_0} A_1 = A|_{E_h}$), where $\mathcal{F}_0$ and $\mathcal{F}_1$ are the projections of the vector $\mathcal{F}$ onto $E_0$ and $E_h$, respectively.

*Proposition* 4. The equation $R\boldsymbol{\phi}_0 = \boldsymbol{\phi}_0$ holds.

Note that the vector $R\phi_0$ is an eigenvector of $A$, corresponding to the zeroth eigenvalue. In fact, in view of the properties of the matrix $R$, we have $AR\phi_0 = -RA\phi_0 = 0$. Hence, $R\phi_0 = a\phi_0$, where $a$ is a certain constant. Multiplying the last equation by $R$ we obtain that $a = \pm 1$. Proposition 4 fixes $a = 1$.

Note that when Proposition 4 is satisfied the equation $R\phi_1 = -\phi_1$ holds. In fact, it follows from the equation $A\phi_1 = \phi_0$ and Proposition 4 that $AR\phi_1 = -\phi_0$, whence we obtain the required result.

If Proposition 4 is satisfied, then in the $\phi_0$, $\phi_1$ basis

$$R|_{E_0} = R_0 = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \tag{1.7}$$

The central part of the solution $v_0$ can be represented in the form $\mathbf{v}_0 = a_0(\xi)\phi_0 + a_1(\xi)\phi_1$. By the theorem of the central manifold [4], for $\mathbf{v}$, which does not leave a fairly small neighbourhood of zero for all $\xi$ and fairly small $\mu$, $\mathbf{v}_h = h(\mu, \mathbf{v}_0)$, where $h(0, 0) = 0$ and $\partial h(0, 0)/\partial \mathbf{v}_0$. In addition, the function $h$ inherits the property of invertibility of system (1.5), namely, $R_h h(\mu, \mathbf{v}_0) = -h(\mu, R_0\mathbf{v}_0)$, where $R_h = R|_{Eh}$.

The problem of investigating the properties of the small solutions (1.5), therefore reduces to investigating the properties of the small solutions of the first system from (1.6), where $\mathcal{F}_0 = \mathcal{F}_0(\mu, \mathbf{v}_0 + h(\mu, \mathbf{v}_0))$. This system is obviously closed, and in the case considered is a second-order system.

The conjugate matrix $A^*$ possesses an eigenvector $\psi_1$ and an associated vector $\psi_0$ for zero eigenvalue. The matrix $A_0$ in the $\phi_0$, $\phi_1$ basis has the form

$$A_0 = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}$$

the vector $\mathbf{v}_0$ in the $\phi_0$, $\phi_1$ basis has the components $a_0$, and $a_1$, while the vector $\mathcal{F}_0$ has the component $f_1 = \langle \mathcal{F}, \psi_0 \rangle$ and $f_2 = \langle \mathcal{F}, \psi_1 \rangle$, where the brackets $\langle \, , \rangle$ denote the usual scalar product in $\mathbf{R}^m$.

*Proposition* 5. For $\mathbf{v}_0 = (a_0, a_1)^t$ sufficiently small, in the expansion $f_2 = \mu\rho_0 a_0 + \rho_1 a_0^2 + \rho_2 a_0 a_1 + \rho_3 a_1^2 + o(|\mu \mathbf{v}_0|, |\mathbf{v}_0|^2)$ in the neighbourhood of $\mathbf{v}_0 = 0$ the constants $\rho_0$ and $\rho_1$ are not equal to zero ($\rho_2 = 0$ by virtue of the invertibility of the equations).

We will prove that the following theorem holds when Proposition 1–5 are satisfied.

*Theorem* 1. The system of equations (1.1) has solutions of the solitary-wave type at least when their amplitude is fairly small.

*Proof.* It follows from Proposition 5, that the first system in (1.6) in the $\phi_0$, $\phi_1$ basis takes the form

$$\begin{aligned} \dot{a}_0 &= a_1 + O(|\mu \mathbf{v}_0|, |\mathbf{v}_0|^2) \\ \dot{a}_1 &= \rho_0 \mu a_0 + \rho_1 a_0^2 + \rho_3 a_1^2 + o(|\mu \mathbf{v}_0|, |\mathbf{v}_0|^2) \end{aligned} \tag{1.8}$$

To investigate system (1.8) further we will use the theory of normal forms [4, 5]. The normal form of system (1.8) can be obtained after making the replacements

$$\mathbf{v}_0 = \boldsymbol{\alpha} + T(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} = (\alpha_0, \alpha_1)^t$$

and is given by the equation

$$\dot{\boldsymbol{\alpha}} = A_0(\boldsymbol{\alpha}) + N(\boldsymbol{\alpha}) + o(|\boldsymbol{\alpha}|^s)$$

where $T(\alpha) = \{T_1(\alpha), T_2(\alpha)\}^t$ and $N(\alpha) = \{N_1(\alpha), N_2(\alpha)\}^t$ are polynomials of power $s$, and $s$ is an arbitrary natural number. The polynomial $N(\alpha)$ satisfies the relations [4]

$$D_\alpha N(\alpha) A_0^* \alpha = A_0^* N(\alpha) \tag{1.9}$$

$$D_\alpha N(\alpha) = \begin{Vmatrix} \partial N_1 / \partial \alpha_0 & \partial N_1 / \partial \alpha_1 \\ \partial N_2 / \partial \alpha_0 & \partial N_2 / \partial \alpha_1 \end{Vmatrix}$$

Further using the invertibility of $N$ with respect to $R_0$ from (1.7): $N(R_0\alpha) = -R_0 N(\alpha)$, and from (1.9) we easily obtain $N_1 = 0$, $N_2 = N_2(\alpha_0)$. The normal form of Eqs (1.8) therefore has the form

$$\dot\alpha_0 = \alpha_1, \quad \dot\alpha_1 = \Phi(\mu, \alpha_0) \tag{1.10}$$

$$\Phi(\mu, \alpha_0) = \sum_{j=1}^{2} c_j(\mu)\alpha_0^j + o(|\mu\alpha_0|, |\alpha_0|^2)$$

The vector polynomial $T(\alpha)$ satisfies the relations [4]

$$D_\alpha T(\alpha) A_0 \alpha = \mathcal{F}_0(\alpha) - N(\alpha)$$

from which it follows that $c_1 = \rho_0\mu + O(\mu^2)$, $c_2 = \rho_1 + O(\mu)$. Then, making the scaling

$$a_0 = \frac{3}{2}\frac{|\rho_0|}{|\rho_1|}\mu\beta_0(\zeta), \quad \alpha_1 = \frac{3}{2}\frac{|\rho_0|}{|\rho_1|}\mu\nu\beta_1(\zeta), \quad \zeta = \nu\xi, \quad \nu = |\rho_0\mu|^{1/2} \tag{1.11}$$

in (1.10) we obtain, apart from terms of the second order of smallness

$$\dot\beta_0 = \beta_1, \quad \dot\beta_1 = \operatorname{sgn}\mu(\operatorname{sgn}\rho_0\beta_0 + \tfrac{3}{2}\operatorname{sgn}\rho_1\beta_0^2) \tag{1.12}$$

Equations (1.12) have soliton-like solutions of the following types:

1. supercritical ($\mu > 0$) solitary waves ($\operatorname{sgn}\rho_0 = 1$)
   (a) $\operatorname{sgn}\rho_1 = -1$—solitary waves of elevation: $\beta_0^* = \operatorname{ch}^{-2}(\zeta/2)$
   (b) $\operatorname{sgn}\rho_1 = 1$—solitary waves of the "well" type: $\beta_0^* = -\operatorname{ch}^{-2}(\zeta/2)$;
2. subcritical ($\mu < 0$) solitary waves ($\operatorname{sgn}\rho_0 = -1$)
   (a) $\operatorname{sgn}\rho_1 = -1$—solitary waves of the "well" type
   (b) $\operatorname{sgn}\rho_1 = -1$—solitary waves of elevation.

To complete the proof of Theorem 1 we will use the following lemma, the proof of which is given, for example, in [1].

*Lemma* 1. Suppose $\mathbf{a}^* = \{\alpha_0^*, \alpha_1^*\}$, where $\alpha_0^*$, $\alpha_1^*$ are related to the solitary-wave type solution $\beta_0^*$, $\beta_1^*$ of system (1.12) by formulae (1.11). Then, for sufficiently small $\mu$, a family of solitary waves $\mathbf{a} = \{a_0, a_1\}^t$ exists, which satisfy the complete system (1.8). In addition, these solutions of the complete system differ only slightly from $\mathbf{a}^*$, namely

$$|\mathbf{a} - \mathbf{a}^*| \leq c_0\mu^2 \exp(-\sigma|\zeta|)$$

where $c_0$ is a certain constant while $\sigma < 1$.

## 2. THE EXISTENCE OF GENERALIZED SOLITARY WAVES

Below, instead of satisfying Proposition 1, we will require that the following proposition should be satisfied.

*Proposition* 1′. A number $i_0$ of the branch $\omega_{i_0}(k) = g_{i_0}(k)$ of the dispersion relation (1.2) exists, such that the order of tangency of the straight line $ck$ ($c = c_{i_0}$) of the graph of the branch $\omega_{i_0}(k)$ for $k = 0$ is two. In addition, the straight line $ck$ has exactly one intersection with the graph of any branch of the dispersion equation when $k > 0$.

In view of the invertibility of the system when $k < 0$ there will also be exactly one intersection of $ck$ and one of the curves $\omega_i(k)$ $(i = 1, \ldots, n)$.

It immediately follows from Proposition 1′ that the essential part of the spectrum of matrix $A$ in (1.5) consists of four eigenvalues, which lie on the imaginary axis. These eigenvalues are a double zero and two non-zero pure imaginary eigenvalues $\pm iq$. The solution $\mathbf{v}_0$ can now be represented in the form

$$\mathbf{v}_0 = a_0\phi_0 + a_1\phi_1 + a_+\phi_+ + a_-\phi_-, \quad \phi_- = \overline{\phi}_+, \quad a_- = \overline{a}_+$$

where $\phi_+$ and $\phi_-$ are the eigenvectors of $A$, corresponding to imaginary eigenvalues $\pm iq$. The vector function $\mathbf{v}_0$ in the $\phi_0$, $\phi_1$, $\phi_0$, $\phi_+$ $\phi_-$ basis has components $a_0$, $a_1$, $a_+$ and $a_-$, while the matrix has the form

$$A_0 = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & iq & 0 \\ 0 & 0 & 0 & -iq \end{Vmatrix}$$

It is clear that the eigenvectors $\phi_+$ and $\phi_-$ can always be chosen so that $R\phi_+ = \phi_-$. In fact, by multiplying $A\phi_+ = iq\phi_+$ by $R$ we obtain $AR\phi_+ = -iqR\phi_+$, whence it follows that $R\phi_+ = a\phi_-$, where $a = \exp(i\theta)$. If $\theta \neq 0$, we choose $\phi_+$ as $\phi_+ = \exp(-i\theta/2)\phi_+$.

In this section we will show that when Propositions 1′ and 2–5 are satisfied, system (1.1) has solutions of the travelling solitary wave type, possibly, with a rapidly oscillating decaying ripple, whose amplitude is much less than the amplitude of the solitary wave itself. The conditions for which a rapidly oscillating ripple must necessarily be present as a component of the generalized solitary wave are not known at the present time.

By virtue of assumption 4 the matrix $R_0$ in the $\phi_0$, $\phi_1$, $\phi_0$, $\phi_+$ $\phi_-$ basis has the form

$$R_0 = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{Vmatrix} \tag{2.1}$$

It follows from (2.1) that the normal form of the first system of equations in (1.6) (which in this case is a system of four equations) has the form [2]

$$\dot{\alpha}_0 = \alpha_1, \quad \dot{z} = iqz + iz\Psi(\mu, \alpha_0, |z|^2)$$
$$\dot{\alpha}_1 = \Phi(\mu, \alpha_0, |z|^2), \quad \dot{\bar{z}} = -iq\bar{z} - i\bar{z}\Psi(\mu, \alpha_0, |z|^2) \tag{2.2}$$

where $\Phi$ and $\Psi$ are real polynomials of arbitrary order $s$ given by

$$\Phi = \sum_{i=1}^{2} c_i(\mu)\alpha_0^i + d_2(\mu)|z|^2 + \ldots, \quad \Psi = \gamma_0(\mu) + \gamma_1(\mu)\alpha_0 + \ldots$$
$$c_1 = \rho_0\mu + O(\mu^2), \quad c_2 = \rho_1 + O(\mu)$$

System (2.2), apart from terms of order up to $O(\mu^s)$, for any $s$ approximates the first system from (1.6) in the neighbourhood of $\mathbf{v}_0 = 0$ and has two first integrals $|z|^2 = K$ and $\alpha_1^2 = \hat{\Phi} = H$, $\partial_{\alpha_0}\hat{\Phi} = 2\Phi$. The constants $d_2$, $\gamma_0$, $\gamma_1$ are defined uniquely from the equations in the polynomial $T$ from Section 1, where $\mathcal{F}_0$ has the component $f_1 = \langle\mathcal{F}_0, \psi_0\rangle$, $f_2 = \langle\mathcal{F}_0, \psi_1\rangle$, $f_3 = \langle\mathcal{F}_0, \psi_+\rangle$, $f_4 = \langle\mathcal{F}_0, \psi_-\rangle$. Here $\langle\, , \rangle$ is the scalar product in $\mathbf{C}^m$, $\psi_1$, $\psi_+ = \overline{\psi}_-$ are eigenvectors and $\psi_0$ is the associated vector of $A^*$. Equation (2.2), apart from terms up to $|\mu\alpha|$, $|\alpha|^2$ inclusive, has the form

$$\dot{\alpha}_0 = \alpha_1, \quad \dot{z} = iz(q + \gamma_0(\mu) + \gamma_2\alpha_0^2 + \gamma_3 z\bar{z})$$
$$\dot{\alpha}_1 = \rho_0\mu\alpha_0 + \rho_1\alpha_0^2; \quad \dot{\bar{z}} = -iz(q + \gamma_0(\mu) + \gamma_2\alpha_0^2 + \gamma_3 z\bar{z}) \tag{2.3}$$

Assuming $z = r \exp i\varphi$, $K = 0$ in (2.3) and making the scaling (1.11) and $\varphi = \phi(\nu\xi)/\nu$, we obtain from (2.3)

$$\partial_\zeta^2 \beta_0 = \operatorname{sgn} \mu(\operatorname{sgn} \rho_0 \beta_0 + \frac{3}{2} \operatorname{sgn} \rho_1 \beta_0^2) + O(\mu)$$
$$\partial_\zeta r = 0, \quad \partial_\zeta \phi = q + O(\mu) \tag{2.4}$$

In the zeroth approximation in $\mu$ system (2.4) has solutions of the solitary-wave type with $r = 0$, described in Section 1. Obviously Eqs (2.2) for $K = 0$ and any order $s$ of the polynomials $\Phi$ and $\Psi$ will have primarily solutions of the solitary-wave type with $z = 0$, which are approximated, apart from terms up to $\mu^2$, by the formulae given in Section 1. Despite the fact that solitary-wave type solutions remain for the system which approximates the complete system (1.6), apart from terms up to any order in $\mu$, they may not be present in the complete system. To clarify this we will formulate two lemmas, which are proved for the types of normal forms considered [2].

We will first consider the constant solution of system (2.4) $\beta_0^* = 0$, $r^* = 0$, where $\beta_0^* = 0$ is a saddle point. This solution will also satisfy system (2.2) for any $s$ and $K = 0$.

*Lemma* 2. Suppose $(\alpha_0^* = 0, \alpha_1^* = 0, z^* = 0\}$ is a solution of system (2.2), where $\Phi$, $\Psi$ are polynomials of arbitrary order $s$. Then, for any sufficiently small $\mu$ a unique periodic solution $\{\tilde{a}_0, \tilde{a}_1, \tilde{a}_+\}$ of the complete system (the first system in (1.6)) exists. Moreover, for a certain constant $C = C(s)$

$$|\{\tilde{a}_0, \tilde{a}_1, \tilde{a}_+\}| \leq C|\mu|^{s-1}$$

uniformly with respect to $\xi$. Similarly

$$|\partial_\xi \varphi(\mu) - \partial_\xi \varphi^*| \leq C_1|\mu|$$

where $\varphi^* = q\xi$, as follows from (2.4).

The last lemma implies that, on changing to a complete system, a periodic solution with an amplitude as small as desired may emerge from the zero solution.

We will further represent the solitary waves $\alpha_0^*$, $\alpha_1^*$, $z^* = 0$, described in Section 1, in the form $\{\gamma^*, 0\}$, where $\gamma^* = \{\alpha_0^*, \alpha_1^*\}$. Suppose $(\tilde{\gamma}, \tilde{a}_+) = \{\tilde{a}_0, \tilde{a}_1, \tilde{a}_+\}$, is the extension of the zero solution to the solution of the complete system (1.6). We then have the following lemma.

*Lemma* 3. For sufficiently small $\mu$ a solution $\{a_0, a_1, a_+\}$ of the first system in (1.6) exists of the generalized solitary-wave type. Moreover, if we write

$$\{a_0, a_1, a_+\} = (\tilde{\gamma}, \tilde{a}_+) + (\underline{\gamma}, \delta \exp(i\varphi))$$

the following limits hold

$$|(\tilde{\gamma}, \tilde{a}_+)| \leq C|\mu|^{s-1}, \quad |\underline{\gamma} - \underline{\gamma}^*| \leq C_1\mu^2 \exp(-\sigma|\zeta|), \quad |\delta| \leq C_2|\mu|^{s-1} \exp(-\sigma|\zeta|)$$

for any $s$, $0 < \sigma < 1$ and certain constants $C$, $C_1$ and $C_2$, where $C_1$ and $C_2$ depend on $s$.

## 3. SOLITARY AND GENERALIZED SOLITARY WAVES IN A COLD PLASMA AND IN EXTENSIBLE RODS

As an example of the system of equations of the type (1.1), which we have proposed and formulated in Section 2, we will consider the equations which describe one-dimensional wave motions in a cold quasi-neutral plasma [6]

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + (\rho + 1)^{-1} \frac{\partial}{\partial x} \frac{B_y^2 + 2\sin\varphi B_y + B_z^2}{2} = 0$$

$$\frac{\partial \upsilon}{\partial t} - (\rho+1)^{-1} B_x \frac{\partial B_y}{\partial x} = -R_e^{-1} \frac{d}{dt}\left\{(\rho+1)^{-1}\frac{\partial B_z}{\partial x}\right\}$$

$$\frac{\partial w}{\partial t} - (\rho+1)^{-1} B_x \frac{\partial B_z}{\partial x} = R_e^{-1} \frac{d}{dt}\left\{(\rho+1)^{-1}\frac{\partial B_y}{\partial x}\right\} \tag{3.1}$$

$$\frac{\partial B_y}{\partial t} - B_x\frac{\partial \upsilon}{\partial x} + (B_y + \sin\varphi)\frac{\partial u}{\partial x} = R_i^{-1}\frac{\partial}{\partial x}\frac{dw}{dt}$$

$$\frac{\partial B_z}{\partial t} - B_x\frac{\partial w}{\partial x} + B_z\frac{\partial u}{\partial x} = -R_i^{-1}\frac{\partial}{\partial x}\frac{d\upsilon}{dt}$$

$$\left(\frac{\partial}{\partial t} \equiv \frac{d}{dt} + u\frac{\partial}{\partial x}\right)$$

where the dependent and independent variables are used in dimensionless form. Here $\varphi$ is the angle of inclination of the vector of the unperturbed magnetic field to the $x$ axis, $\rho$ is the perturbation of the particle density, $B_y$ and $B_z$ are the corresponding components of the perturbed magnetic field, $B_x = \cos\varphi$, $(u, \upsilon, w)$ are the components of the velocity vector of the ionic fluid, and $R_i$ and $R_e$ are dispersion parameters characterizing the Larmor frequency of the ions and electrons.

The two branches of the dispersion relation for (3.1) have the form [6]

$$\omega_\pm(k) = \frac{k}{2(1+R_i^{-1}R_e^{-1}k^2)}\{X_+ \pm X_-\}$$

$$X_\pm = [(1\pm\cos\varphi)^2 + \{Y \pm 2\cos\varphi\}R_i^{-1}R_e^{-1}k^2]^{1/2}$$

$$Y = (R_e R_i^{-1} + R_i R_e^{-1})\cos^2\varphi + \sin^2\varphi$$

where $\omega_+(k)$ corresponds to the magnetoacoustic branch while $\omega_-(k)$ corresponds to the Alfvén branch. The velocity $c = \lim_{k\to 0}\omega_+/k = 1$ is the phase velocity of infinitely long magneto-acoustic waves.

System (3.1) for travelling waves $(\rho, u, \upsilon, w, B_y, B_z) = \mathbf{w}(x - Vt)$ can be represented in the form (1.3), where

$$G(\mathbf{w}) = diag\left(1, \frac{V-u}{V}, \frac{V-u}{V,}\ \frac{V-u}{V,}, 1, 1\right),\ V = c + \mu$$

while the expressions for $L$ and $\mathbf{f}$ follow in an obvious way from (1.1). The matrix $G$ is invertible in the neighbourhood of $\mathbf{w} = 0$, and the system corresponding to system (1.5) can be represented in the following explicit form relative to the leading derivatives with respect to $\xi$

$$\dot{\upsilon} = -R_i B_z - \frac{R_i\cos\varphi}{V}(\rho+1)w$$

$$\dot{w} = \frac{R_i\cos\varphi}{V}(\rho+1)\upsilon + R_i B_y - R_i\sin\varphi\rho \tag{3.2}$$

$$\dot{B}_y = R_e(\rho+1)w + \frac{R_e\cos\varphi}{V}(\rho+1)B_z$$

$$\dot{B}_z = -R_e(\rho+1)\upsilon - \frac{R_e\cos\varphi}{V}(\rho+1)B_y$$

The quantities $u$ and $\rho$ can be expressed in terms of $\mathbf{v} = \{\upsilon, w, B_y, B_z\}^t$ algebraically

$$u = \frac{1}{2V}(B_y^2 + 2\sin\varphi B_y + B_z^2),\ \rho = \frac{u}{V-u}$$

Hence, it is obvious that system (4.1) satisfies Propositions 2 and 3. The inversion matrix for (3.2) is given by the formula

$$R = \mathrm{diag}(1, -1, 1, -1)$$

Henceforth we will assume that $0 < \varphi \leqslant \pi/2$. The matrix $A$ from (1.5) in this case has the form

$$A = \begin{Vmatrix} 0 & -R_i \cos\varphi & 0 & -R_i \\ R_i \cos\varphi & 0 & R_i \cos^2\varphi & 0 \\ 0 & R_e & 0 & R_e \cos\varphi \\ -R_e & 0 & -R_e \cos\varphi & 0 \end{Vmatrix}$$

We will further consider two cases of qualitatively different behaviour of the dispersion relation of Eqs (3.1).

The case $\varphi > \varphi_c$, $\varphi_c = \mathrm{arctg}(\sqrt{(R_e R_i^{-1} - R_i R_e^{-1})})$.

In this case Proposition 1 is satisfied for the straight line $\omega = k$. The corresponding graphs of the magnetoacoustic and Alfvén branches are presented in Fig. 1, where we show the mutual position of the straight line $\omega = k$ and the graph of the dispersion relation of the cold plasma. Curve 3 corresponds to the Alfvén branch, curve 2 corresponds to the magnetoacoustic branch, while the straight line 1 is the graph of $\omega = k$. Straight line 1 has a tangency of the second order to curve 2 at $k = 0$. In view of this, the secular equation for $A$ has only one root, situated on the imaginary axis—a second-order zero. The eigenvector $\phi_0$ and the associated vector $\phi_1$, corresponding to the zeroth eigenvalue, are

$$\phi_0 = \begin{Vmatrix} -\cos\varphi \\ 0 \\ 1 \\ 0 \end{Vmatrix}, \quad \phi_1 = \begin{Vmatrix} 0 \\ R_e^{-1} - R_i^{-1}\cos^2\varphi \\ 0 \\ \left(R_i^{-1} - R_e^{-1}\right)\cos\varphi \end{Vmatrix} \tag{3.3}$$

Obviously, $R\phi_0 = \phi_0$ and Proposition 4 is therefore satisfied. The eigenvector $\psi_1$ and the associated vector $\psi_0$ of matrix $A^*$ have the form

$$\psi_0 = R_i^{-1}\Delta^{-1} \begin{Vmatrix} (R_i^{-1} - R_e^{-1})\cos\varphi \\ 0 \\ R_i R_e^{-1}(R_1^{-1} - R_e^{-1}\cos^2\varphi) \\ 0 \end{Vmatrix}, \quad \psi_1 = R_i^{-1}\Delta^{-1} \begin{Vmatrix} 0 \\ 1 \\ 0 \\ R_i R_e^{-1}\cos\varphi \end{Vmatrix}$$

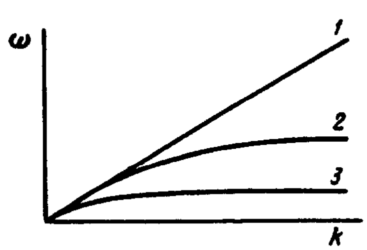


Fig. 1.

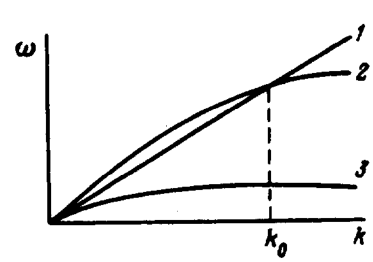


Fig. 2.

which implies the equality (apart from terms of the third order of smallness)

$$f_2 =< \mathcal{T},\ \psi_1 >= \Delta^{-1}\left\{\mu a_0 - \frac{3}{2}\sin\varphi a_0^2 - \frac{1}{2}(R_e^{-1} - R_i^{-1})^2 \cos^2\varphi \sin\varphi a_1^2\right\} \tag{3.4}$$

where $\Delta = R_e^{-1}R_i^{-1}\cos^2\varphi(\mathrm{tg}^2\varphi - \mathrm{tg}^2\varphi_c)$, while $\mathcal{F}$, up to terms of the order of $O(|\mu v_0|, |v_0|^2)$ inclusive, has the form

$$\mathcal{F} = \begin{Vmatrix} \frac{\mu}{2}R_i\cos\varphi w - R_i\cos\varphi\sin\varphi w B_y \\ -\frac{\mu}{2}R_i\cos\varphi v + \mu R_i\sin^2\varphi B_y + \frac{R_i}{2}\sin 2\varphi v B_y - \frac{R_i}{2}\sin\varphi(B_y^2 + B_z^2) - R_i\sin^3\varphi B_y \\ -\frac{\mu}{2}R_e\cos\varphi B_z + R_e\cos\varphi\sin\varphi B_y B_z + R_e\sin\varphi w B_y \\ \frac{\mu}{2}R_e\cos\varphi B_y - R_e\sin\varphi v B_y - R_e\cos\varphi\sin\varphi B_y^2 \end{Vmatrix}$$

The correctness of Proposition 5 follows from (3.4) with $\rho_0 = \Delta^{-1}$ and $\rho_1 = -3/2\sin\varphi\,\Delta^1$.

The existence of a family of solitary waves for sufficiently small $\mu = V - 1 > 0$ therefore follows from the results of Section 2. These solitary waves will be waves of elevation

$$a_0 = \frac{\mu}{\sin\varphi\,\mathrm{ch}^2(|\mu|^{1/2}\Delta^{-1/2}\xi)} + O(\mu^2)$$

The case $\mathrm{tg}^2\varphi < \mathrm{tg}^2\varphi_c$.

In this case the dispersion curves have the form shown in Fig. 2 and the straight line $\omega = k$ satisfies Proposition 1′. The vectors $\phi_0$ and $\phi_1$, as before, are given by (3.3), while

$$\phi_+ = \begin{Vmatrix} R_i q^{-1}(1 - R_i R_e^{-1}\cos^2\varphi) \\ iR_i R_e^{-1}\cos\varphi \\ q^{-1}(R_i - R_e)\cos\varphi \\ -i \end{Vmatrix},\quad \phi_- = \bar{\phi}_+$$

In Fig. 2 we show the mutual position of the straight line $\omega = k$ and the graph of the dispersion relation for the cold plasma for $\varphi < \varphi_c$ and $k > 0$. Straight line 1, in addition to a second-order tangency at the point $k = 0$ to curve 2, has an additional intersection with this curve at $k_0 > 0$. It follows from the results of Section 2, that in this case there are generalized solitary waves of the "well" type (in the case considered $\Delta < 0$ and $\mu < 0$) with oscillations whose amplitude is less than $C|\mu|^s$ ($C(s)$ is a certain constant) for any $s$. Here, we have not ruled out the possibility that the amplitude of these oscillations is equal to zero.

We will further discuss an example in which Proposition 4 breaks down. Consider system (3.2) in the region of the Alfvén velocity $c = \cos\varphi$. The straight line $ck$ in this case has a second-order tangency to the graph of the Alfvén branch 3 of the dispersion relation and intersects the magnetoacoustic branch at $k > 0$ (Fig. 3), i.e. Proposition 1′ is satisfied. The matrix $A$ has the form

$$A = \begin{Vmatrix} 0 & -R_i & 0 & -R_i \\ R_i & 0 & R_i(1-\mathrm{tg}^2\varphi & 0 \\ 0 & R_e & 0 & R_e \\ -R_e & 0 & -R_e & 0 \end{Vmatrix}$$

and its eigenvectors and associated vector are given by the formulae

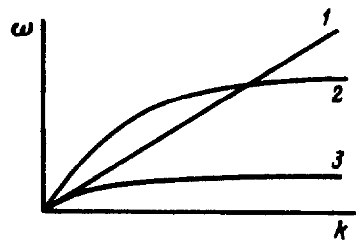


Fig. 3.

$$\phi_0 = \left\| \begin{matrix} 0 \\ -1 \\ 0 \\ 1 \end{matrix} \right\|, \quad \phi_1 = \left\| \begin{matrix} R_e^{-1}(1-\mathrm{tg}^2\,\varphi) - R_i^{-1} \\ 0 \\ R_i^{-1} - R_e^{-1} \\ 0 \end{matrix} \right\|$$

$$\phi_+ = \left\| \begin{matrix} -1 \\ iq^{-1}[R_i - R_e(1-\mathrm{tg}^2\,\varphi)] \\ R_e R_i^{-1} \\ -iq^{-1}R_e(1-R_i^{-1}R_e) \end{matrix} \right\|, \quad \phi_- = \overline{\phi}_+.$$

It is easy to see that $R\phi_0 = -\phi_0$. This indicates formally that Eqs (2.3) cannot be obtained, i.e. in the polynomials $\Phi$ and $\Psi$ a dependence of $\alpha_1$ is allowed, which also occurs in this case. In view of this, the approximating equations have no solutions of the generalized solitary-wave type. This explains why attempts to derive the non-linear evolution equation (of the Korteweg–de Vries type with a cubic non-linearity) for long Alfvén waves of small amplitude have not been successful [6]. Note that the Korteweg–de Vries equation was introduced into modern plasma physics by Sagdeyev in 1964 [7].

To conclude this section we will consider the propagation of longitudinal waves in plane extensible elastic rods. We will assume that the stretching and bending of the rod occur in the $x_1x_2$ plane and that the rod, in the undeformed state, coincides with the $x_1$ axis. The energy density of this rod is made up of the densities of the kinetic energy $K$, the flexural energy $C$ and the compression energy $\Pi$, where

$$K = \frac{1}{2}\rho S x_t^i x_{it}, \quad B = \frac{1}{2} I E x_{ss}^i x_{iss}, \quad \Pi = \frac{1}{8} E(x_s^i x_{is} - 1)^2$$

Here $\rho$ is the density of the rod, $E$ is Young's modulus of the material, $S$ is the cross-sectional area of the rod, $\rho I$ is the moment of inertia of the cross-section about an axis passing through the centre of inertia of the cross-section and perpendicular to the $x_1x_2$ plane, and $s$ is the length of the arc along the elastic line of rod; the subscript $i$ takes values of 1 and 2, summation is assumed over repeated indices ($x^i = x_i$) and the subscripts $t$ and $s$ denote differentiation with respect to the corresponding variable. After the scaling

$$x_i \to \sqrt{I}x_i, \quad t \to \sqrt{\frac{\rho S I}{E}}t, \quad s \to \sqrt{I}s$$

the ratio of the Lagrangian of the system to Young's modulus $E$ can be written in the form

$$\frac{1}{2}\int_{t_0}^{t}\int_{-\infty}^{\infty} \{x_t^i x_{it} - x_{ss}^i x_{iss} - \frac{1}{4}(x_s^i x_{is} - 1)^2\} ds$$

In canonical variables $x_s^i = \tau_\infty^i + \tau^i, x_t^i = \upsilon^i$, where $\tau_i$ are the components of the perturbations of the tangential vector to the elastic line of the rod, $\upsilon_i$ are the components of the velocity of points of the elastic line and $\tau^i_\infty = \delta^i_1$, the system of equations describing wave propagation in the rod is a Hamiltonian system and has the form

$$\tau^i_t = \upsilon^i_s,\ i = 1,2$$
$$\upsilon_{1t} = \frac{1}{2}\tau_{1s} - \tau_{1sss} + \frac{1}{4}\{3\tau_1^2 + \tau_2^2 + \tau_1^3 + \tau_1\tau_2^2\}_s \tag{3.5}$$
$$\upsilon_{2t} = -\tau_{2sss} + \frac{1}{4}\{2\tau_1\tau_2 + \tau_1^2\tau_2 + \tau_2^3\}_s$$

The dispersion equation of system (3.5)

$$(\omega^2 - k^4)(\omega^2 - \frac{1}{2}k^2 - k^4) = 0 \tag{3.6}$$

satisfies Proposition 1′, the dispersion curve for $k > 0$ is shown in Fig. 4 (curves 2 and 3), and straight line 1 corresponds to $\omega = 1/\sqrt{(2k)}$.

Further, we will put $\xi = s - Vt$. The system of equations (3.5) also obviously satisfies Propositions 2 and 3 with the unique matrix $G(V, \mathbf{w})$, ($\mathbf{w} = (\tau_i, \upsilon_i)'$, $i = 1, 2$), and the equations for travelling waves in expanded form in terms of leading derivatives can be written in the form in terms of leading derivatives can be written in the form

$$\dot{\tau}_i = u_i,\ i = 1,2$$
$$\dot{u}_1 = -\mu\tau_1 + \frac{1}{4}(3\tau_1^2 + \tau_2^2 + \tau_1^3 + \tau_1\tau_2^2)$$
$$\dot{u}_2 = -\frac{1}{2}\tau_2 - \mu\tau_2 + \frac{1}{4}(2\tau_1\tau_2 + \tau_1^2\tau_2 + \tau_2^3)\ (\mu = V^3 - 1/2)$$

$$A = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \end{Vmatrix},\quad R = \mathrm{diag}(1, -1, -1, 1)$$

where the dot denotes differentiation with respect to $\xi$ and the eigenvectors and associated vector of the matrix $A$ are given by


Fig. 4.

$$\phi_0 = \begin{Vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Vmatrix}, \quad \phi_1 = \begin{Vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Vmatrix}, \quad \phi_+ = \begin{Vmatrix} 0 \\ i \\ 0 \\ -1/\sqrt{2} \end{Vmatrix}$$

It can be seen that Proposition 4 in the case considered is satisfied. The eigenvector $\psi_1$ of the conjugate matrix $A^*$ is identical with $\phi_1$, while

$$\psi_+ = \frac{1}{2} \begin{Vmatrix} 0 \\ i \\ 0 \\ \sqrt{2} \end{Vmatrix}$$

and $\rho_0 = 1$, $\rho_1 = 3/4$, whence it follows that Proposition 5 is satisfied. As a result we obtain the relation

$$\tau_1 = -2\mu \, \mathrm{ch}^{-2} \frac{\sqrt{\mu}}{2} \xi + O(\mu^2), \quad \mu > 0$$

which describes compression waves in the rod. In the equation for $a_+$ the function $f_3$ (see Section 2) is identically equal to zero. This means that $a_+ = 0$ and, consequently, the amplitude of the periodic component is equal to zero (this case is not a common situation) and compression of the rod in this case is not accompanied by bending.

## 4. DISCUSSION OF THE RESULTS

Propositions 1–5 and 1′ have a specific meaning. Thus, Proposition 1 contains the condition that a wave having a phase velocity of $c_{i_0}$ in the linear approximation does not interact with any other wave (the graph of not a single branch of the dispersion equation intersects the straight line $c_{i_0}k$). Proposition 1′ contains the condition that in the linear approximation we have resonance with the periodic wave corresponding to wave number $k_0$, where the straight line $c_{i_0}k$ intersects one of the branches of the dispersion relation.

The solution of the complete problem can then be written in the form

$$\mathbf{v} = a_0^* \phi_0 + a_1^* \phi_1 + a_+^* \phi_+ + a_-^* \phi_- + O(\mu^2)$$

where $\phi_0$, $\phi_+ = \bar{\phi}_-$ are the eigenvectors and $\phi_1$ is the associated vector of the matrix $A$ from (1.5), corresponding to the central spectrum of $A$, and

$$a_0^* = \pm \frac{3}{2} \mu \frac{|\rho_0|}{|\rho_1|} \mathrm{ch}^{-2} \frac{|\rho_0 \mu|^{1/2}}{2} \xi + O\left(\mu^2 \exp\left(-\sigma \frac{|\rho_0 \mu|^{1/2}}{2} \xi\right)\right), \quad 0 < \sigma < 1$$

(the choice of the plus or minus sign in the last formula depends on the sign of $\rho_0$, $\rho_1$ and $\mu$, as discussed in Section 1), and

$$a_1^* = \frac{\partial}{\partial \xi} a_0^* + O(\mu^s), \quad a_+^* + a_-^* = r \cos(q\xi + \delta) + o(r), \quad r = O(\mu^s)$$

for any $s$. The phase of the $\delta$-bounded function and $\delta(\infty) - \delta(-\infty) \neq 0$ [2], which indicates the existence of a phase shift in the periodic component of the solution. The conditions for the function $r$ to be strictly non-zero are unknown. Moreover, there are examples when both $r \neq 0$ and $r = 0$ when Proposition 1′ is satisfied.

1. Consider the equation

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}+\frac{\partial^5 u}{\partial x^5}=0$$

which describes, in particular, long surface waves of small amplitude in a liquid with surface effects [8–10]. Proposition 1′ is satisfied for this equation. For $\mu = V - c > 0$ we have generalized solitary waves with a periodic ripple with a non-zero exponentially small amplitude [11]. In addition, there is a rigorous proof of the fact that there are no solutions of the solitary-wave type with a ripple of zero amplitude for the range of velocities considered.

2. The well-known Hiroti–Ito equation

$$\frac{\partial u}{\partial t}+a\left(6n\frac{\partial u}{\partial x}+u\frac{\partial^3 u}{\partial x^3}\right)+b\left(45u^2\frac{\partial u}{\partial x}+15u\frac{\partial^3 u}{\partial x^3}+15\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^5 u}{\partial x^5}\right)=0$$

which is also subject to Proposition 1′ when $a > 0$ and $b > 0$, gives an example when actual solitary waves exist when $V > 0$, which means that there is no periodic component.

Propositions 2, 3 and 5 have a common form and are related to the structure of Eqs (1.1). Proposition 4 is related to the conservation of invertibility in the following sense. The presence of invertibility in system (1.5) means the expectation that inverse solutions $\mathbf{v}(\xi)$ exist, part of the components of which are even functions and part odd. This formally implies the existence of a diagonal matrix $R$ (inversion) with components $\pm 1$ on the diagonal, such that the following equation is satisfied

$$R\mathbf{v}(-\xi)=\mathbf{v}(\xi) \tag{4.1}$$

We recall that for small solutions $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_h$ the function $\mathbf{v}_h = h(\mu, \mathbf{v}_0)$ possesses corresponding hereditary symmetries, so that for inverse solutions to exist it is sufficient to indicate the solution of system (1.6) $\mathbf{v}_0$, which satisfies Eq. (4.1).

In accordance with Proposition 4 the following equation follows from (4.1)

$$\begin{aligned}R\mathbf{v}_0(-\xi)&=a_0(-\xi)R\phi_0+a_1(-\xi)R\phi_1+a_+(-\xi)R\phi_++a_-(-\xi)R\phi_-=\\&=a_0(-\xi)\phi_0-a_1(-\xi)\phi_1+a_+(-\xi)\phi_-+a_-(-\xi)\phi_+=\\&=\mathbf{v}_0(\xi)=a_0(\xi)\phi_0+a_1(\xi)\phi_1+a_+(\xi)\phi_++a_-(\xi)\phi_-\end{aligned} \tag{4.2}$$

It follows from (4.2) that $a_0$ is an even function, $a_1$ is an odd function, and $a_+$ and $a_-$ have an even real part and an odd imaginary part (these properties of $a_+$ and $a_-$ can always be achieved, as already pointed out, by an appropriate choice of the vectors $\phi_\pm$). These properties of the coefficients $a_0$, $a_1$, $a_+$, $a_-$ are obviously necessary for even solutions to exist, which the solitary and generalized solitary waves considered in fact are.

I wish to thank A. G. Kulikovskii for discussions and support.

This research was carried out with financial support from the Russian Foundation for Basic Research (93-013-17355) and the International Science Foundation (MDM000).

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*Translated by R.C.G.*